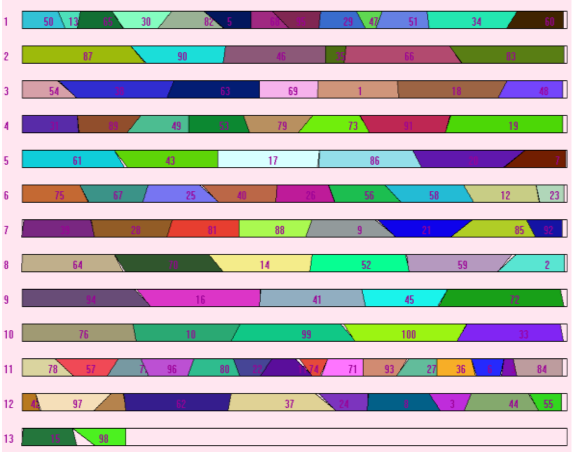
Bin Packing with Trapezia: Methods and Applications



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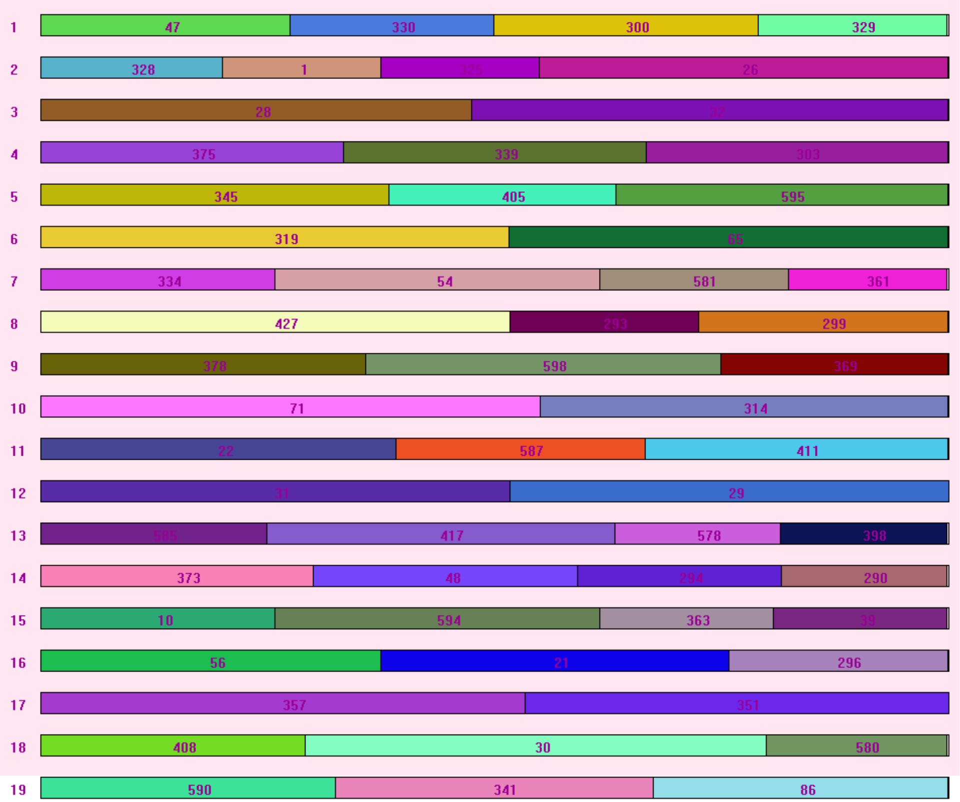
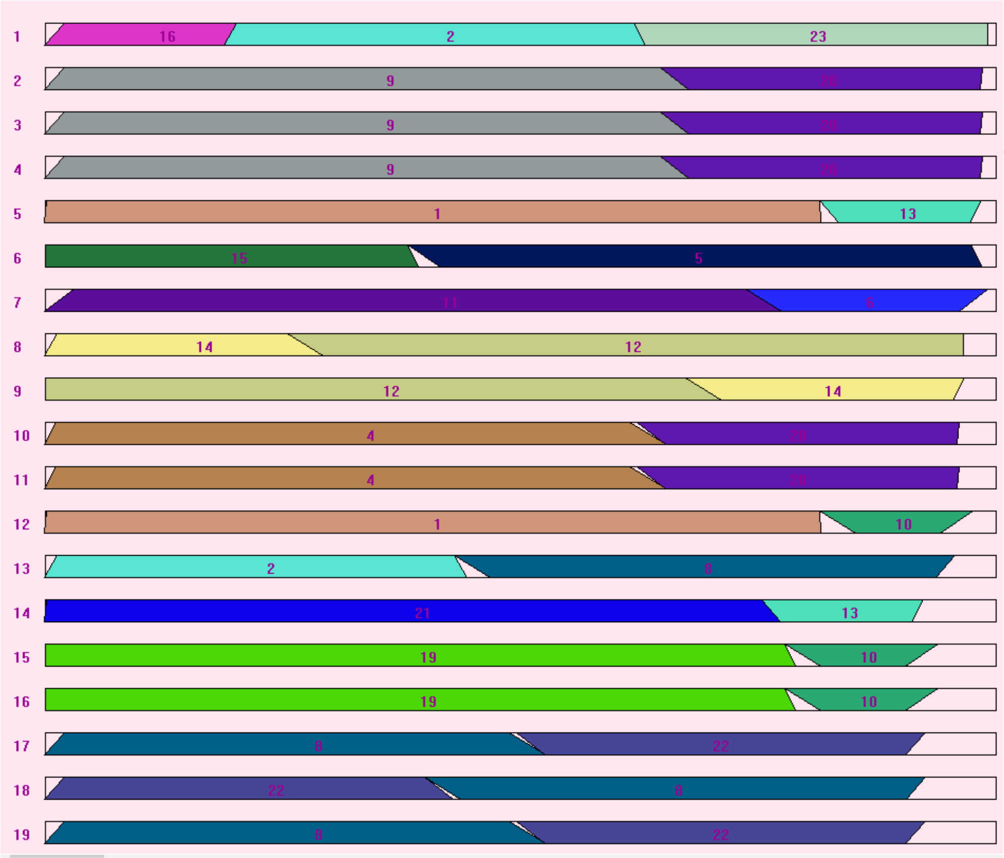
The 1D Bin Packing Problem



- In a standard 1D bin packing problem we seek to pack 1D items into a minimum number of fixed-sized bins.
- It is NP-complete (generalises the partitioning problem).
- The order and orientation of items within each group irrelevant
- But what if it is?

The Trapezium Packing Problem



Problem:

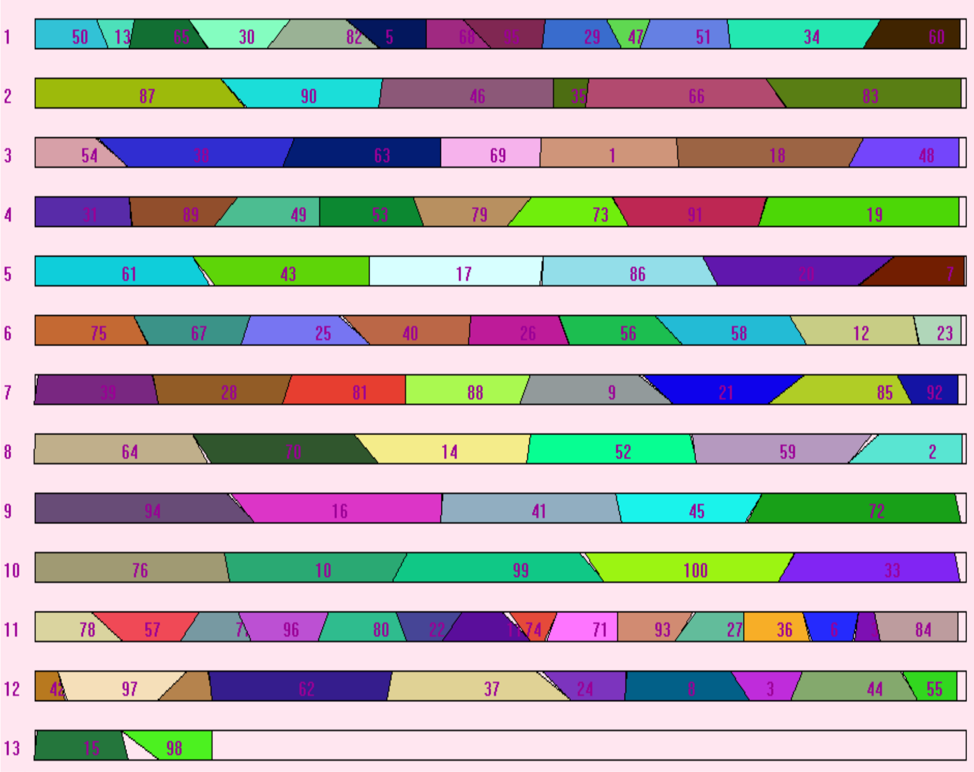
Arrange a set of fixed-height trapezia into a minimal number of bins

Issues

1) Which items should be put into the same bin? (A bin packing problem)

2) How should items be arranged in a bin? (An ordering and rotation problem)

The Trapezium Packing Problem



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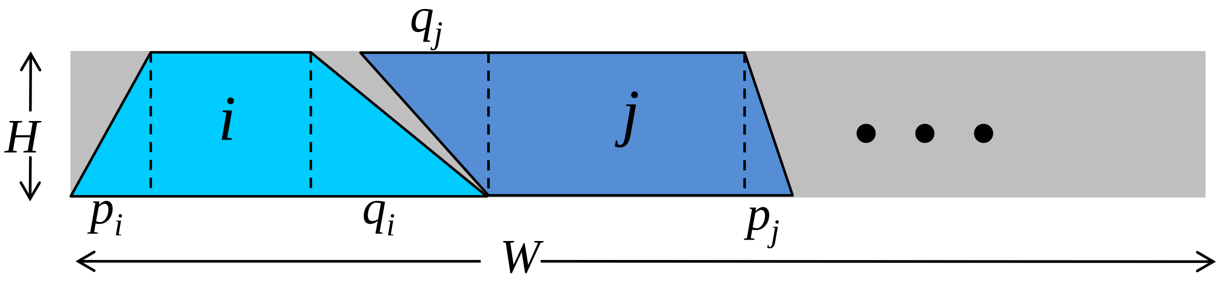
Real World Applications

Arranging the trapezoids in a single bin



- Items have 4 orientations, but we only need to consider two
- Inter-item wastage in the above is simply $|q_i - q_j|$
- Wastage w = all inter-item wastage plus the LHS and RHS

- Problem Def: Given a group of items S such that:

Total area of trapezoids in the group

$$A(S) = \sum_{j \in S} A(j) \le HW$$

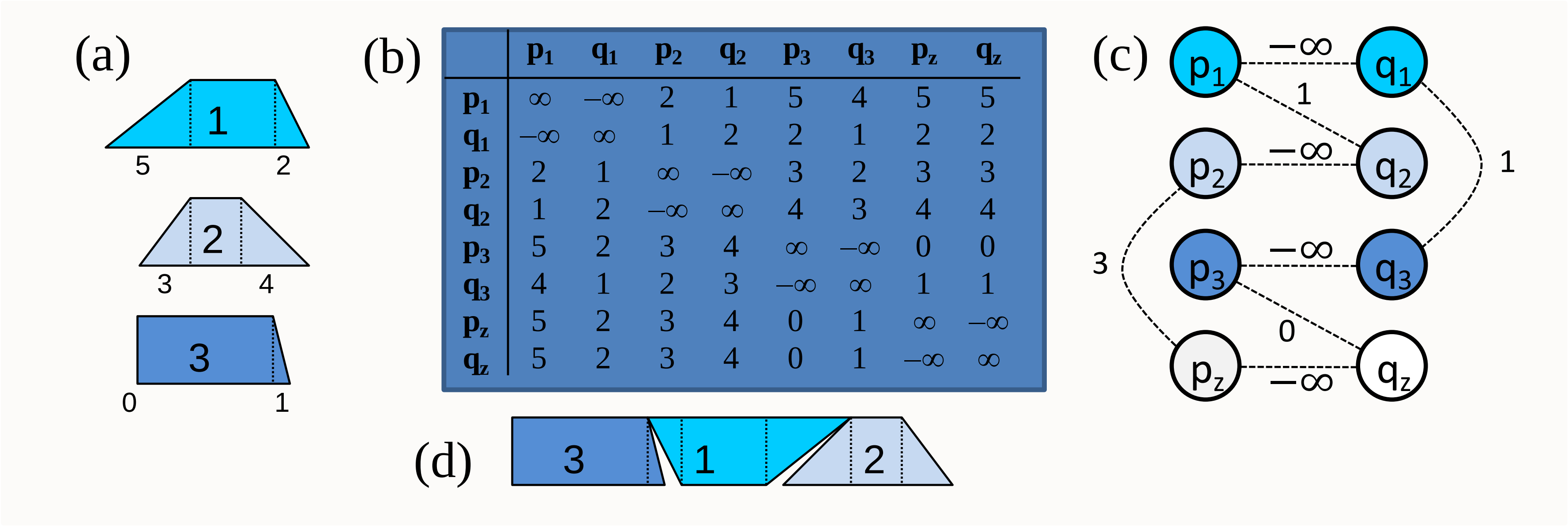
Area of the bin

can we determine an item arrangement such that:

$$\underbrace{A(S) + w}_{\text{Total area of trapezoids in the group plus inter-item wastage}} \le HW$$

TSP interpretation of the single bin problem

- Arranging trapezoids in a single bin can be seen as a special type of TSP
- Each projection is a “city” ; edges between cities correspond to wastage; and edges on the same shape are set to minus infinity
- Can a “valid” route of less than *w* be achieved (discounting minus infinity arcs)?



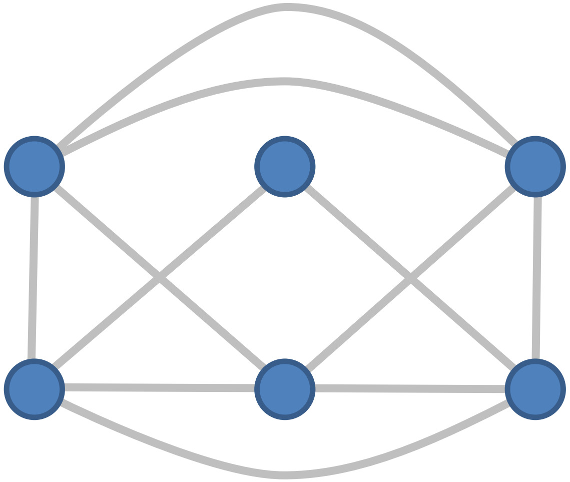
	p_1	q_1	p_2	q_2	p_3	q_3	p_z	q_z
p_1	∞	$-\infty$	2	1	5	4	5	5
q_1	$-\infty$	∞	1	2	2	1	2	2
p_2	2	1	∞	$-\infty$	3	2	3	3
q_2	1	2	$-\infty$	∞	4	3	4	4
p_3	5	2	3	4	∞	$-\infty$	0	0
q_3	4	1	2	3	$-\infty$	∞	1	1
p_z	5	2	3	4	0	1	∞	$-\infty$
q_z	5	2	3	4	0	1	$-\infty$	∞

Revision: Eulerian Graphs and Cycles

- **Def:** An Eulerian cycle is a cycle that uses every edge in a graph exactly once.

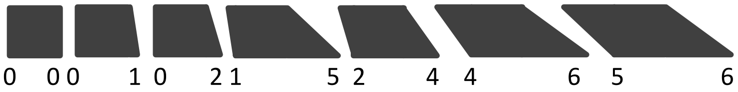
- **Def:** An Eulerian graph is a graph that is connected and where all vertex degrees are even

- **Fact:** A graph has an Eulerian cycle if and only if it is Eulerian.



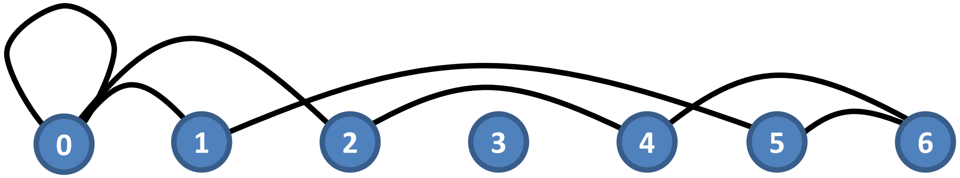
How to Solve the Single Bin Problem

- Take a set of trapezoids



- Form a multigraph G using projection pairs as edges

E = { {0,0}, {0,1}, {0,2}, {1,5}, {2,4}, {4,6}, {5,6} }



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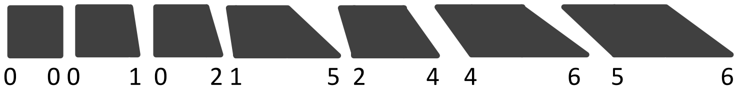

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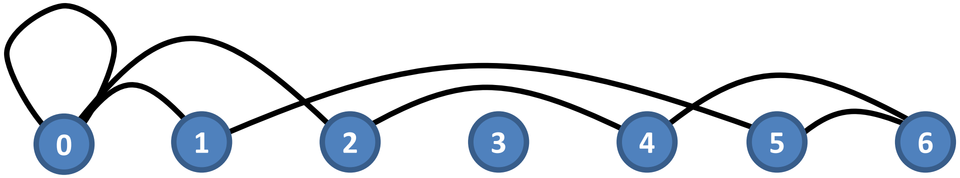
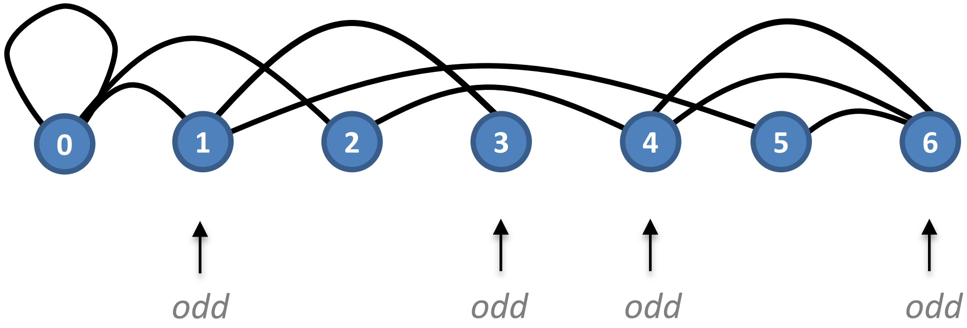


- **Fact:** If G is Eulerian, then a zero cost arrangement exists.
- This is the case here: ((0,0), (0,1), (1,5), (5,6), (6,4), (4,2), (2,0))

How to Solve the Single Bin Problem

- **Fact.** If G has odd-degree vertices, then a minimum weight matching can be introduced to make them even.
- For example,



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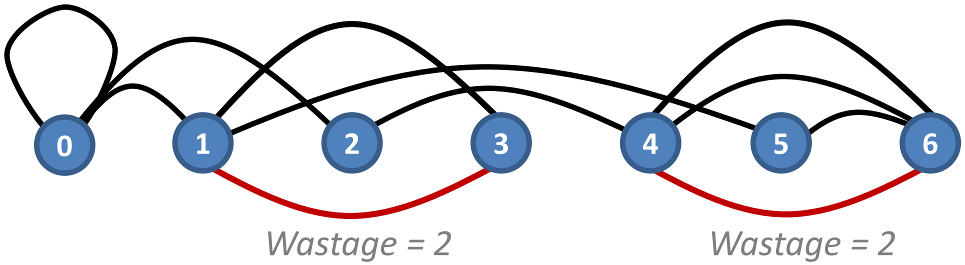

- If G is now connected, an optimal arrangement is an Eulerian cycle in this new graph as before

((0,0),(0,2),(2,4),(4,6),(6,4),(6,5),(5,1),(1,3),(1,0))

Wastage = 2 *Wastage = 2*

How to Solve the Single Bin Problem

- If all degrees are even but G is disconnected, the components need to be joined
- **Fact.** Each of the n Eulerian components is optimal.
- **Fact.** Optimally joining two components gives us n – 1 optimal Eulerian components

w = 4

- ((1,1),(1,3),(3,5),(5,7),(7,6),(6,2),(2,1)) *Wastage = 0*
- ((8,8)) *Wastage = 0*
- ((9,9),(9,11),(11,13),(13,15),(15,13),(15,14),(14,10),(10,12),(10,9)) *Wastage = 4*

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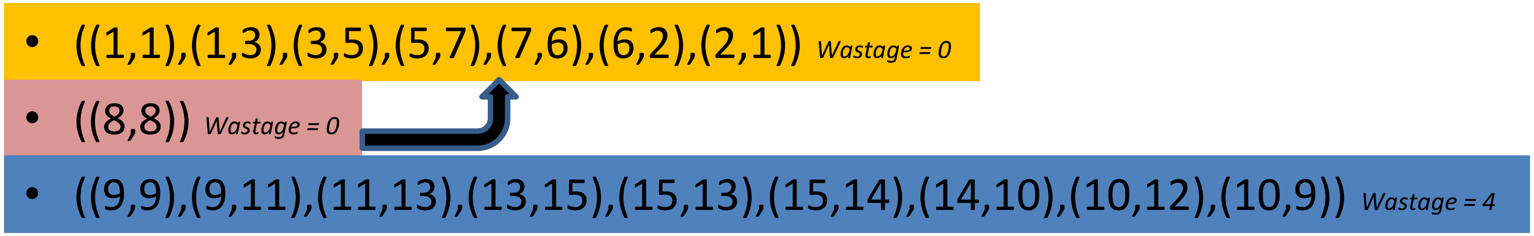

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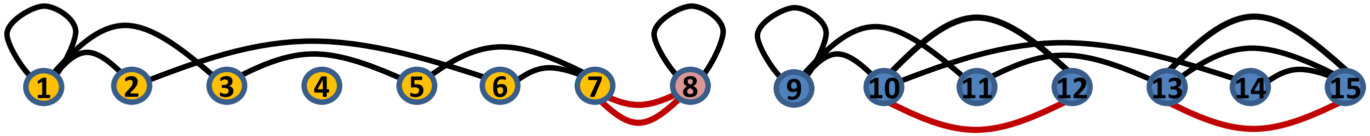

w = 6

- ((1,1),(1,3),(3,5),(5,7), (8,8) ,(7,6),(6,2),(2,1)) *Wastage = 2*

- ((9,9),(9,11),(11,13),(13,15),(15,13),(15,14),(14,10),(10,12),(10,9)) *Wastage = 4*

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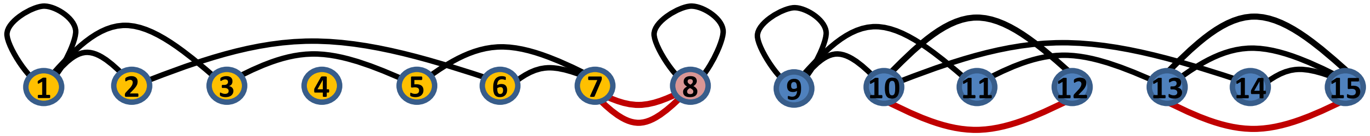

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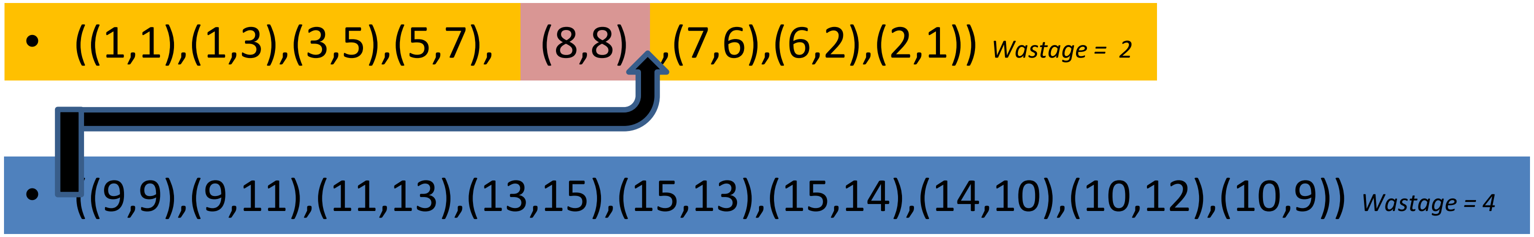

- ((1,1),(1,3),(3,5),(5,7), (8,8) ,(7,6),(6,2),(2,1)) *Wastage = 2*
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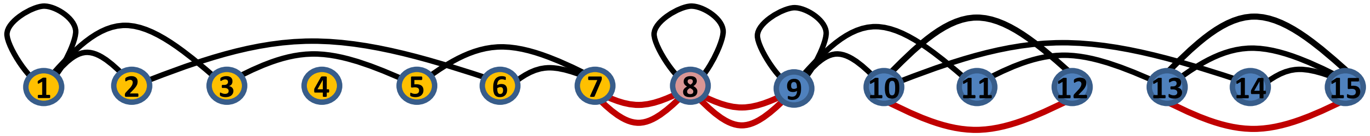
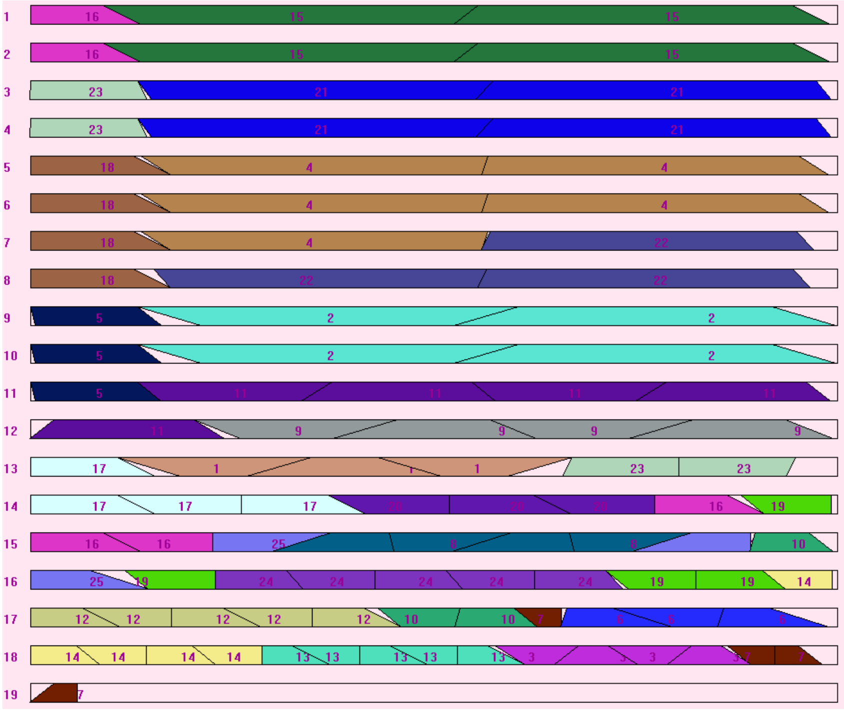


w = 8

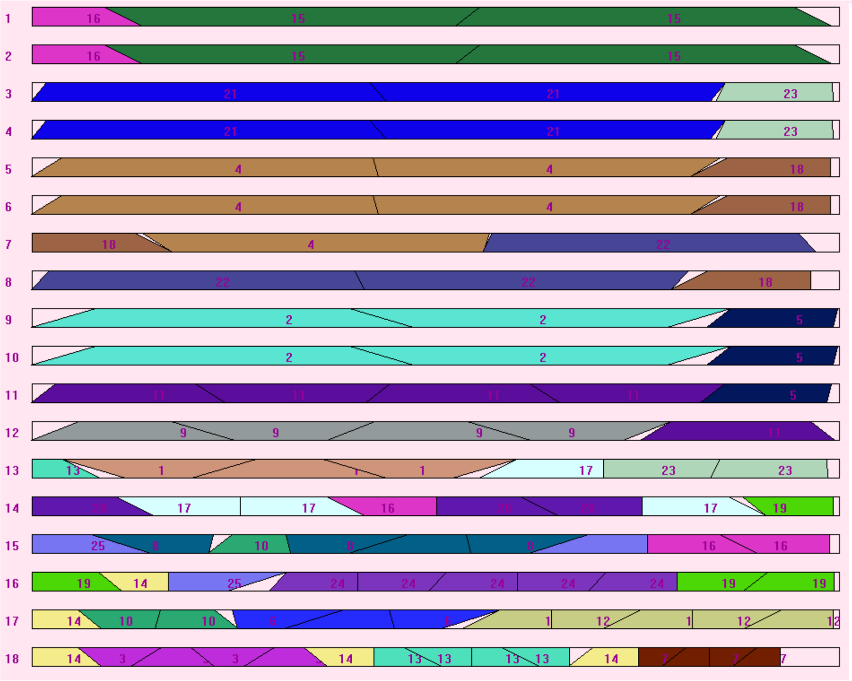
- ((1,1),(1,3),(3,5),(5,7), (8,8) (9,9),(9,11),(11,13),(13,15),(15,13), (15,14),(14,10),(10,12),(10,9) ,(7,6),(6,2),(2,1)) *Wastage = 8*

- A proof naturally follows which demonstrates this method to be exact. The problem using multiple bins is still NP-hard, however.

The multi-bin problem: Comparison using FFD



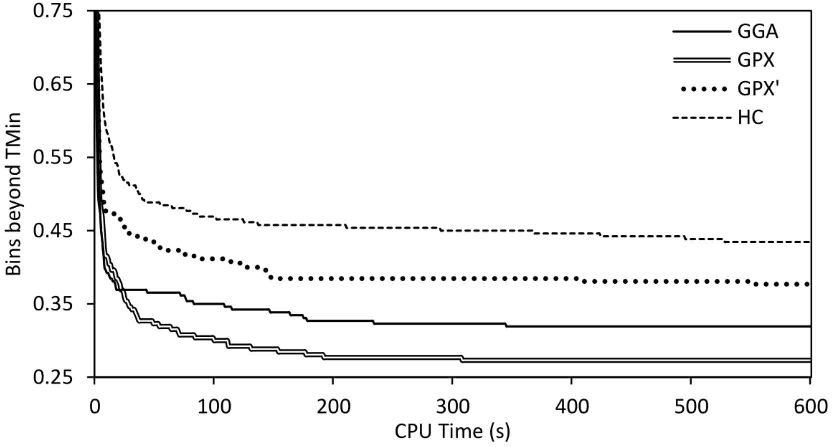
- Example solution using a simple (inexact) heuristic to pack trapezia into individual bins, combined with FFD.



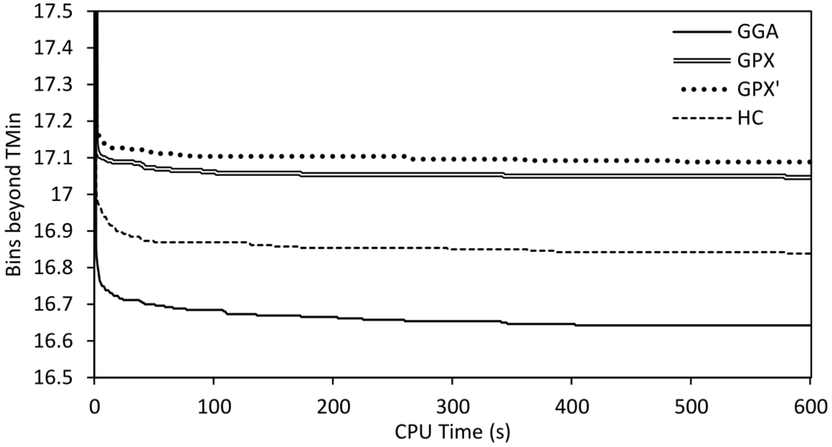
- Solution using the exact algorithm for packing trapezia into individual bins, combined with FFD

Further Improvements via Evolutionary Methods and Local Search

Run profiles for various different evolutionary operators using 500-item problems (averaged across 240 instances)



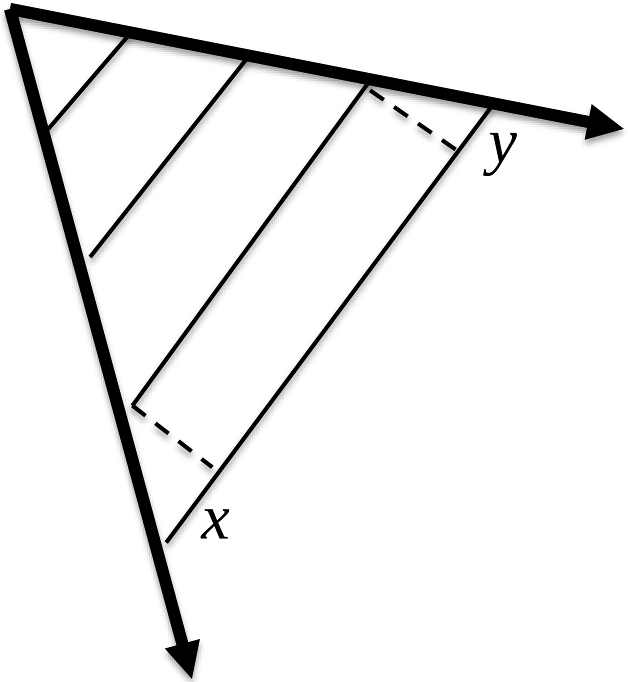
Instances with approximately 9 items per bin

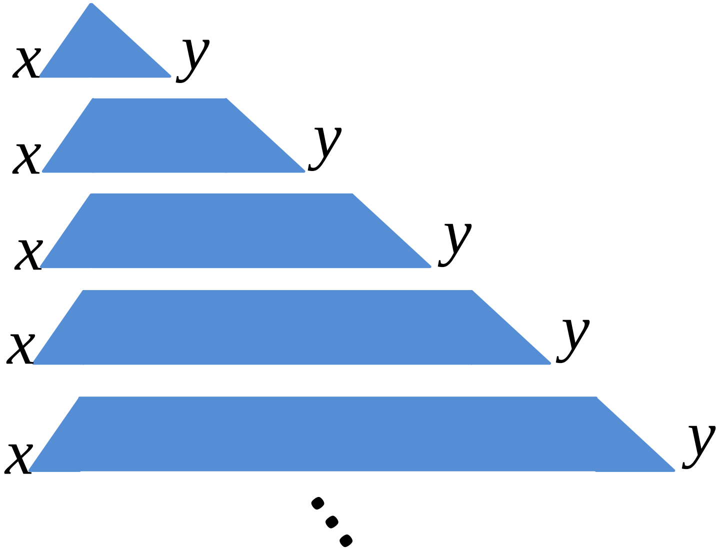


Instances with approximately 2.5 items per bin

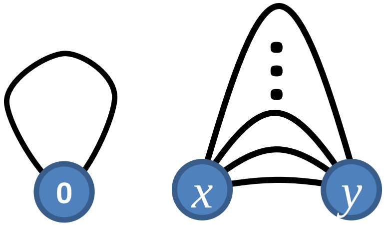
- **Lewis, R.** and P. Holborn (2017) 'How to Pack Trapezoids: Exact and Evolutionary Algorithms'. *IEEE Transactions on Evolutionary Computation*, vol. 21(3), pp. 463-476.

Special Cases Arising in Industry





If item sizes in a 1D BPP are 1, 2, 3, …, j, then a perfect packing exists **if and only if** the sum of all item sizes is a multiple of the bin capacity



- Coffman, E. et al. “Perfect Packing Theorems and the Average-Case Behaviour of Optimal and Online Bin Packing” SIAM Review 2002 44:1, 95-108

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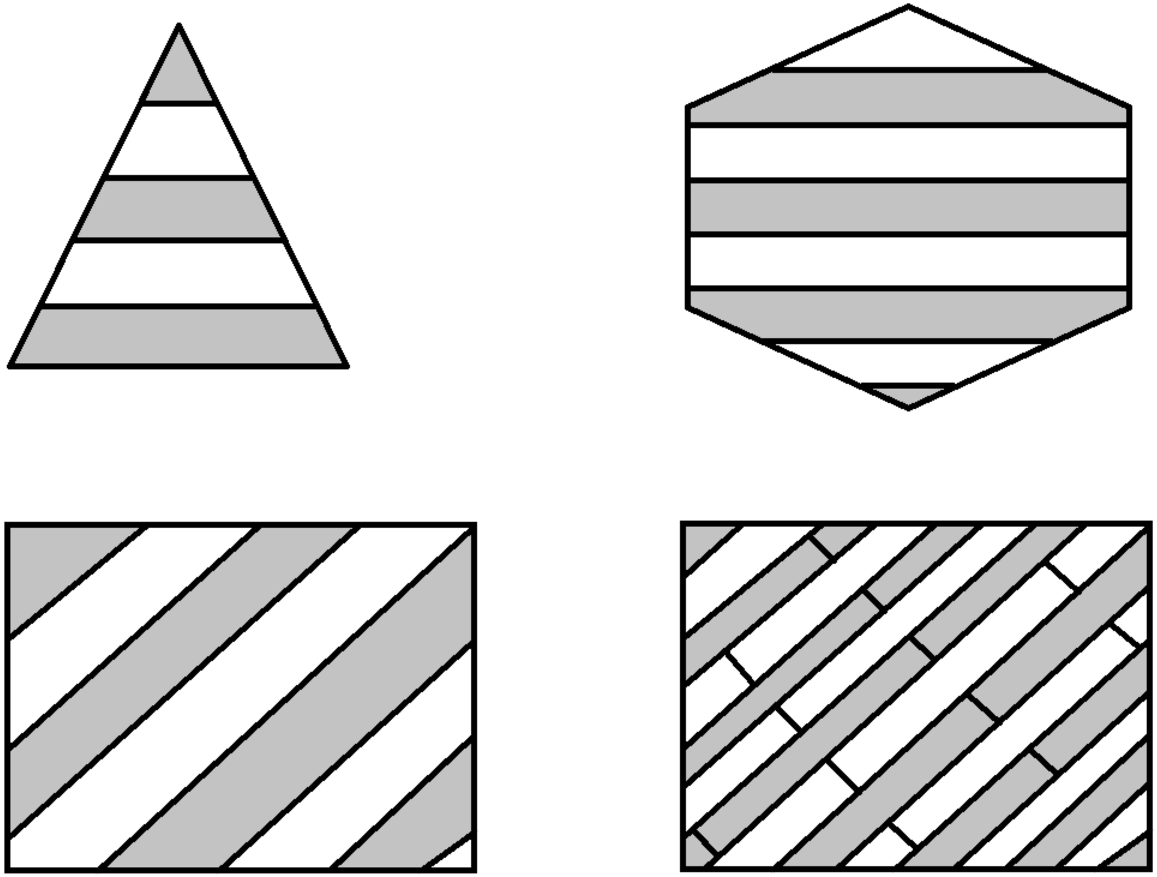

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Questions...

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